

On the Core of a Unicyclic Graph

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Abstract

A set $S \subseteq V$ is *independent* in a graph $G = (V, E)$ if no two vertices from S are adjacent. By $\text{core}(G)$ we mean the intersection of all maximum independent sets. The *independence number* $\alpha(G)$ is the cardinality of a maximum independent set, while $\mu(G)$ is the size of a maximum matching in G .

A connected graph having only one cycle, say C , is a *unicyclic graph*. In this paper we prove that if G is a unicyclic graph of order n and $n - 1 = \alpha(G) + \mu(G)$, then $\text{core}(G)$ coincides with the union of cores of all trees in $G - C$.

Keywords: independent set, core, matching, unicyclic graph, König-Egerváry graph.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X . By $G - W$ we mean the subgraph $G[V - W]$, if $W \subset V(G)$. For $F \subset E(G)$, by $G - F$ we denote the partial subgraph of G obtained by deleting the edges of F , and we use $G - e$, if $W = \{e\}$. If $A, B \subset V$ and $A \cap B = \emptyset$, then (A, B) stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$. The neighborhood of a vertex $v \in V$ is the set $N(v) = \{w : w \in V \text{ and } vw \in E\}$, and $N(A) = \cup\{N(v) : v \in A\}$, $N[A] = A \cup N(A)$ for $A \subset V$. By C_n, K_n we mean the chordless cycle on $n \geq 4$ vertices, and respectively the complete graph on $n \geq 1$ vertices.

A set S of vertices is *independent* if no two vertices from S are adjacent, and an independent set of maximum size will be referred to as a *maximum independent set*. The *independence number* of G , denoted by $\alpha(G)$, is the size of a maximum independent set of G . Let $\Omega(G)$ denote the family $\{S : S \text{ is a maximum independent set of } G\}$, while

$$\text{core}(G) = \cap\{S : S \in \Omega(G)\} \text{ [11].}$$

An edge $e \in E(G)$ is α -critical whenever $\alpha(G - e) > \alpha(G)$. Notice that the inequalities $\alpha(G) \leq \alpha(G - e) \leq \alpha(G) + 1$ hold for each edge e .

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one covering all vertices of G . An edge $e \in E(G)$ is μ -critical provided $\mu(G - e) < \mu(G)$.

Theorem 1.1 [13] *For every graph G no α -critical edge has an endpoint in $N[\text{core}(G)]$.*

It is well-known that

$$\lfloor n/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n$$

hold for every graph G with n vertices. If $\alpha(G) + \mu(G) = n$, then G is called a *König-Egerváry graph* [4], [19]. Several properties of König-Egerváry graphs are presented in [6], [9], [10], [12], [14], [16].

It is known that every bipartite graph is a König-Egerváry graph as well [5], [8]. This class includes also non-bipartite graphs (see, for instance, the graph G in Figure 1).

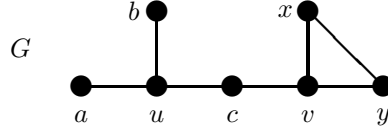


Figure 1: A König-Egerváry graph with $\alpha(G) = |\{a, b, c, x\}|$ and $\mu(G) = |\{au, cv, xy\}|$.

Theorem 1.2 *If G is a König-Egerváry graph, then*

- (i) [12] *every maximum matching matches $N(\text{core}(G))$ into $\text{core}(G)$;*
- (ii) [13] *$H = G - N[\text{core}(G)]$ is a König-Egerváry graph with a perfect matching and each maximum matching of H can be enlarged to a maximum matching of G .*

The graph G is called *unicyclic* if it is connected and has a unique cycle, which we denote by $C = (V(C), E(C))$. Let

$$N_1(C) = \{v : v \in V(G) - V(C), N(v) \cap V(C) \neq \emptyset\},$$

and $T_x = (V_x, E_x)$ be the tree of $G - xy$ containing x , where $x \in N_1(C)$, $y \in V(C)$.

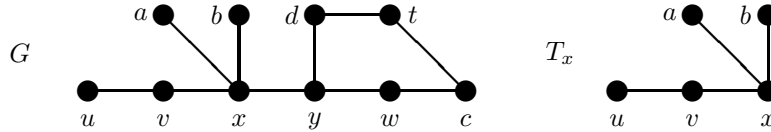


Figure 2: G is a unicyclic non-König-Egerváry graph with $V(C) = \{y, d, t, c, w\}$.

Unicyclic graphs keep enjoying plenty of interest, as one can see, for instance, in [1], [3], [7], [15], [18], [20], [21].

In this paper we analyze the structure of $\text{core}(G)$ for a unicyclic graph G .

2 Results

If G is a unicyclic graph, then there is an edge $e \in E(C)$, such that $\mu(G - e) = \mu(G)$, because for each pair of edges, consecutive on C , at most one could be μ -critical. Let us mention that $\alpha(G) \leq \alpha(G - e) \leq \alpha(G) + 1$ holds for each edge $e \in E(G)$. Every edge of the unique cycle could be α -critical; e.g., the graph G from Figure 2, which has also additional α -critical edges (e.g., the edge uv).

Let us notice that the bipartite graph T_x from Figure 2 has only two maximum matchings, namely, $M_1 = \{ax, uv\}$ and $M_2 = \{bx, uv\}$, while each vertex of $\text{core}(T_x) = \{a, b\}$ is not saturated by one of these matchings.

Lemma 2.1 *For every bipartite graph G , a vertex $v \in \text{core}(G)$ if and only if there exists a maximum matching that does not saturate v .*

Proof. Since $v \in \text{core}(G)$, it follows that $\alpha(G - v) = \alpha(G) - 1$. Consequently, we have

$$\alpha(G) + \mu(G) - 1 = |V(G)| - 1 = |V(G - v)| = \alpha(G - v) + \mu(G - v)$$

which implies that $\mu(G) = \mu(G - v)$. In other words, there is a maximum matching in G not saturating v .

Conversely, suppose that there exists a maximum matching in G that does not saturate v . Since, by Theorem 1.2(i), $N(\text{core}(G))$ is matched into $\text{core}(G)$ by every maximum matching, it follows that $v \notin N(\text{core}(G))$.

Assume that $v \notin \text{core}(G)$. By Theorem 1.2(ii), every maximum matching M of G is of the form $M = M_1 \cup M_2$, where M_1 matches $N(\text{core}(G))$ into $\text{core}(G)$, while M_2 is a perfect matching of $G - N[\text{core}(G)]$. Thus v is saturated by every maximum matching of G , in contradiction with the hypothesis on v . ■

Remark 2.2 *Lemma 2.1 fails for non-bipartite König-Egerváry graphs; e.g., every maximum matching of the graph G from Figure 1 saturates $c \in \text{core}(G) = \{a, b, c\}$.*

Lemma 2.3 *If G is a unicyclic graph of order n , then $n - 1 \leq \alpha(G) + \mu(G) \leq n$.*

Proof. If $e = xy \in E(C)$, then $G - e$ is a tree, because G is connected. Hence, $\alpha(G - e) + \mu(G - e) = n$. Clearly, $\alpha(G - e) \leq \alpha(G) + 1$, while $\mu(G - e) \leq \mu(G)$. Consequently, we get that

$$n = \alpha(G - e) + \mu(G - e) \leq \alpha(G) + \mu(G) + 1,$$

which leads to $n - 1 \leq \alpha(G) + \mu(G)$. The inequality $\alpha(G) + \mu(G) \leq n$ is true for every graph G . ■

Remark 2.4 *If G has n vertices, p connected components, say $H_i, 1 \leq i \leq p$, and each component contains only one cycle, then one can easily see that $n - p \leq \alpha(G) + \mu(G) \leq n$, because $\alpha(G) = \sum_{i=1}^p \alpha(H_i)$ and $\mu(G) = \sum_{i=1}^p \mu(H_i)$.*

While $C_{2k}, k \geq 2$, has no α -critical edge at all, each edge of every odd cycle $C_{2k-1}, k \geq 2$, is α -critical. This property is partially inherited by unicyclic graphs.

Lemma 2.5 *Let G be a unicyclic graph of order n . Then $n - 1 = \alpha(G) + \mu(G)$ if and only if each edge of its unique cycle is α -critical.*

Proof. Assume that $n - 1 = \alpha(G) + \mu(G)$. Since G is connected, for each $e \in E(C)$ the graph $G - e$ is a tree. Hence, we have

$$\alpha(G - e) - \alpha(G) + \mu(G - e) - \mu(G) = 1,$$

which implies $\mu(G - e) = \mu(G)$ and $\alpha(G - e) = \alpha(G) + 1$, since

$$-1 \leq \mu(G - e) - \mu(G) \leq 0 \leq \alpha(G - e) - \alpha(G) \leq 1.$$

In other words, every $e \in E(C)$ is α -critical.

Conversely, let $e \in E(C)$ be such that $\mu(G - e) = \mu(G)$; such an edge exists, because no two consecutive edges on C could be μ -critical. Since e is α -critical, and $G - e$ is a tree, we infer that

$$n - 1 = \alpha(G - e) + \mu(G - e) - 1 = \alpha(G) + \mu(G),$$

and this completes the proof. ■

Combining Lemma 2.5 and Theorem 1.1, we infer the following.

Corollary 2.6 *If G is a unicyclic non-König-Egerváry graph, then no vertex of its unique cycle belongs to $N[\text{core}(G)]$.*

Remark 2.7 *Corollary 2.6 is true also for some unicyclic König-Egerváry graphs; e.g., the graph H_1 from Figure 3. However, the König-Egerváry graph H_2 from the same figure satisfies $N[\text{core}(H_2)] \cap V(C) = \{u\} \neq \emptyset$.*

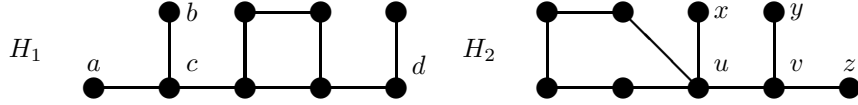


Figure 3: H_1 and H_2 have $N[\text{core}(H_1)] = \{a, b, c\}$, $N[\text{core}(H_2)] = \{x, y, z, u, v\}$.

Lemma 2.8 *Let G be a unicyclic graph of order n . If there exists some $x \in N_1(C)$, such that $x \in \text{core}(T_x)$, then G is a König-Egerváry graph.*

Proof. Let $x \in \text{core}(T_x)$, $y \in N(x) \cap V(C)$, and $z \in N(y) \cap V(C)$. Suppose, to the contrary, that G is not a König-Egerváry graph. By Lemmas 2.3 and 2.5, the edge yz is α -critical. Hence $y \notin \text{core}(G)$, which implies that $\alpha(G) = \alpha(G - y)$. In accordance with Lemma 2.1, there exists a maximum matching M_x of T_x not saturating x . Combining M_x with a maximum matching of $G - y - T_x$ we get a maximum matching M_y of $G - y$. Hence $M_y \cup \{xy\}$ is a matching of G , which results in $\mu(G) \geq \mu(G - y) + 1$. Therefore, using Lemma 2.3 and having in mind that $G - y$ is a forest of order $n - 1$, we get the following contradiction

$$n - 1 = \alpha(G) + \mu(G) \geq \alpha(G - y) + \mu(G - y) + 1 = n - 1 + 1 = n,$$

that completes the proof. ■

Remark 2.9 The converse of Lemma 2.8 is not generally true; e.g., the graph H_1 from Figure 3 is a unicyclic König-Egerváry graph, while both $c \notin \text{core}(T_c) = \{a, b\}$, and $d \notin \text{core}(T_d) = \emptyset$.

Theorem 2.10 If G is a unicyclic non-König-Egerváry graph, then

$$\text{core}(G) = \cup \{ \text{core}(T_x) : x \in N_1(C) \}.$$

Proof. *Claim 1.* Every maximum independent set of T_x may be enlarged to some maximum independent set of G , for each $x \in N_1(C)$.

Let $A \in \Omega(T_x)$, $y \in N(x) \cap V(C)$, and $z \in N(y) \cap V(C)$. According to Lemma 2.5, the edge yz is α -critical. Hence there exist $S_y \in \Omega(G)$, $S_{yz} \in \Omega(G - yz)$, such that $y \in S_y$ and $y, z \in S_{yz}$.

Case 1. Assume that $x \notin A$.

If $|S_y - V(T_x)| < \alpha(G - T_x) = |S_0|$, where $S_0 \in \Omega(G - T_x)$, then the set $S_1 = S_0 \cup (S_y \cap V(T_x))$ is independent in G , and we get the contradiction

$$\alpha(G) = |S_y - V(T_x)| + |S_y \cap V(T_x)| < |S_0| + |S_y \cap V(T_x)| = |S_1|.$$

Therefore, we have $|S_y - V(T_x)| = \alpha(G - T_x)$. Then $A \cup (S_y - V(T_x)) \in \Omega(G)$, otherwise we obtain the following contradiction

$$|S_y - V(T_x)| + |A| < \alpha(G) \leq \alpha(G - T_x) + \alpha(T_x) = |S_y - V(T_x)| + |A|.$$

Case 2. Assume now that $x \in A$.

Then we have $|A| \geq |S_{yz} \cap V(T_x)|$, because $S_{yz} \cap V(T_x)$ is independent in T_x . Hence we infer

$$\begin{aligned} \alpha(G) &= |S_{yz} - \{y\}| \leq |(S_{yz} - \{y\} - (S_{yz} \cap V(T_x))) \cup A| = \\ &= |(S_{yz} - \{y\} - V(T_x)) \cup A|. \end{aligned}$$

Since $W = (S_{yz} - \{y\} - V(T_x)) \cup A$ is independent and its size is $\alpha(G)$ at least, it follows that W is also a maximum independent set, i.e., we have $A \subseteq W \in \Omega(G)$, as needed.

Claim 2. $S \cap V(T_x) \in \Omega(T_x)$ for every $S \in \Omega(G)$ and each $x \in N_1(C)$.

Let $S \in \Omega(G)$, and suppose, to the contrary, that $A = S \cap V(T_x) \notin \Omega(T_x)$. By Lemma 2.8, $x \notin \text{core}(T_x)$. Thus we can change A for some $B \in \Omega(T_x)$ not containing x . The set $(S - A) \cup B$ is clearly independent in G , and this leads to the contradiction $|(S - A) \cup B| = |S - A| + |B| > |S| = \alpha(G)$.

Combining *Claims 1* and *2*, we infer that:

$$\begin{aligned} \text{core}(T_x) &= \cap \{A : A \in \Omega(T_x)\} = \cap \{S \cap V(T_x) : S \in \Omega(G)\} \\ &= (\cap \{S : S \in \Omega(G)\}) \cap V(T_x) = \text{core}(G) \cap V(T_x), \end{aligned}$$

which clearly implies

$$\text{core}(G) = \cup \{ \text{core}(T_x) : x \in N(V(C)) - V(C) \}$$

as required. ■

Remark 2.11 *The assertion in Theorem 2.10 may fail for:*

(i) *bipartite unicyclic graphs; for example, the graphs H_1, H_2 from Figure 4 satisfy*

$$\begin{aligned} \text{core}(H_1) &= \cup \{ \text{core}(T_x) : x \in N_1(C) \}, \text{ and} \\ \text{core}(H_2) &\neq \{x, z\} = \cup \{ \text{core}(T_x) : x \in N_1(C) \}; \end{aligned}$$

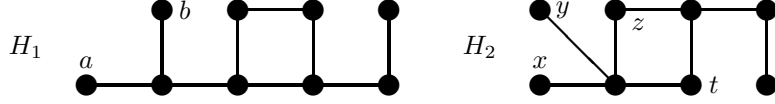


Figure 4: H_1, H_2 are bipartite unicyclic graphs, $\text{core}(H_1) = \{a, b\}$, $\text{core}(H_2) = \{t, x, y, z\}$.

(ii) *non-bipartite König-Egerváry unicyclic graphs; for instance,*

$$\begin{aligned} \text{core}(G_2) &\neq \{t, z\} = \cup \{ \text{core}(T_x) : x \in N_1(C) \}, \text{ while} \\ \text{core}(G_1) &= \cup \{ \text{core}(T_x) : x \in N_1(C) \}, \end{aligned}$$

where G_1 and G_2 are from Figure 5.

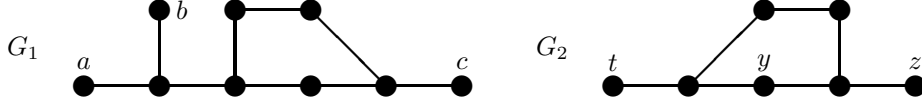


Figure 5: G_1, G_2 are König-Egerváry graphs, $\text{core}(G_1) = \{a, b, c\}$, $\text{core}(G_2) = \{t, y, z\}$.

It is worth mentioning that the problem of whether there are vertices in a given graph G belonging to $\text{core}(G)$ is **NP**-hard [2]. In [17] we have presented both sequential and parallel algorithms finding $\text{core}(G)$ in polynomial time for König-Egerváry graphs. By Theorem 2.10, a unicyclic graph is either a König-Egerváry graph or its $\text{core}(G)$ equals a union of cores of a finite number of some special subtrees. Therefore, we get the following.

Corollary 2.12 *If G is a unicyclic graph, then $\text{core}(G)$ is computable in polynomial time.*

3 Conclusions

The main purpose of this paper is to investigate the structure of $\text{core}(G)$ for unicyclic graphs. On the one hand, we have succeeded to represent $\text{core}(G)$ as the union of cores of some specific subtrees of a non König-Egerváry unicyclic graph G . On the other hand, it is still not clear if there exists a characterization of this kind for bipartite unicyclic graphs and/or non-bipartite König-Egerváry graphs.

References

- [1] F. Belardo, M. Li, M. Enzo, S. K. Simić, J. Wang, *On the spectral radius of unicyclic graphs with prescribed degree sequence*, Linear Algebra and its Applications **432** (2010) 2323-2334.
- [2] E. Boros, M. C. Golumbic, V. E. Levit, *On the number of vertices belonging to all maximum stable sets of a graph*, Discrete Applied Mathematics **124** (2002) 17-25.
- [3] Z. Du, B. Zhou, N. Trinajstić, *Minimum sum-connectivity indices of trees and unicyclic graphs of a given matching number*, Journal of Mathematical Chemistry **47** (2010) 842-855.
- [4] R. W. Deming, *Independence numbers of graphs - an extension of the König-Egerváry theorem*, Discrete Mathematics **27** (1979) 23-33.
- [5] E. Egerváry, *On combinatorial properties of matrices*, Matematikai Lapok **38** (1931) 16-28.
- [6] F. Gavril, *Testing for equality between maximum matching and minimum node covering*, Information Processing Letters **6** (1977) 199-202.
- [7] B. Huo, S. Ji, X. Li, *Note on unicyclic graphs with given number of pendent vertices and minimal energy*, Linear Algebra and its Applications **433** (2010) 1381-1387.
- [8] D. König, *Graphen und Matrizen*, Matematikai Lapok **38** (1931) 116-119.
- [9] E. Korach, T. Nguyen, B. Peis, *Subgraph characterization of red/blue-split graphs and König-Egerváry graphs*, Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, ACM Press (2006) 842-850.
- [10] C. E. Larson, *The critical independence number and an independence decomposition*, European Journal of Combinatorics **32** (2011) 294-300.
- [11] V. E. Levit, E. Mandrescu, *Combinatorial properties of the family of maximum stable sets of a graph*, Discrete Applied Mathematics **117** (2002) 149-161.
- [12] V. E. Levit, E. Mandrescu, *On α^+ -stable König-Egerváry graphs*, Discrete Mathematics **263** (2003) 179-190.
- [13] V. E. Levit, E. Mandrescu, *On α -critical edges in König-Egerváry graphs*, Discrete Mathematics **306** (2006) 1684-1693.
- [14] V. E. Levit, E. Mandrescu, *A characterization of König-Egerváry graphs using a common property of all maximum matchings*, (2009) arXiv:0911.4626 [cs.DM], 9 pp.
- [15] V. E. Levit, E. Mandrescu, *Greedoids on vertex sets of unicycle graphs*, Congressus Numerantium **197** (2009) 183-191.
- [16] V. E. Levit, E. Mandrescu, *Critical independent sets and König-Egerváry graphs*, Graphs and Combinatorics (2011) (accepted), arXiv:0906.4609 [math.CO], 8 pp.

- [17] V. E. Levit, E. Mandrescu, *An algorithm computing the core of a König-Egerváry graph*, (2011) arXiv:1102.1141 [cs.DM], 8 pp.
- [18] J. Li, J. Guo, W. C. Shiu, *The smallest values of algebraic connectivity for unicyclic graphs*, Discrete Applied Mathematics **158** (2010) 1633-1643.
- [19] F. Sterboul, *A characterization of the graphs in which the transversal number equals the matching number*, Journal of Combinatorial Theory Series B **27** (1979) 228-229.
- [20] Y. Wu, J. Shu, *The spread of the unicyclic graphs*, European Journal of Combinatorics **31** (2010) 411-418.
- [21] M. Zhai, R. Liu, J. Shu, *Minimizing the least eigenvalue of unicyclic graphs with fixed diameter*, Discrete Mathematics **310** (2010) 947-955.